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FINITE STRAINS AND DISPLACEMENTS OF ELASTIC MEMBRANES BY THE FINITE ELEMENT METHOD

J. T. ODEN

Engineering Mechanics Department, University of Alabama Research Institute, Huntsville

and

T. SATO

Structures and Mechanics Department, Chrysler Corporation, Huntsville

Abstract—This paper presents a consistent finite element formulation for the analysis of large displacements and finite strains in elastic membranes of general shape. A continuous membrane is divided into a number of flat triangular elements and the behavior of a typical element is described in terms of the displacements of its nodes. It is assumed that the node points are sufficiently close that the displacement fields within each element can be approximated by linear functions of the local coordinates. On the basis of this assumption, the Lagrangian strain tensor is expressed in terms of the node displacements and a nonlinear stiffness relation between node forces and displacements is derived. Group transformations are introduced which re-assemble the elements and apply appropriate boundary conditions. These lead to systems of nonlinear algebraic equations in the generalized displacements. Numerical examples are included to demonstrate the procedure.

NOTATION

Indicial notation and the summation convention are used throughout this paper. Unless otherwise specified, upper-case Latin indices indicate points in space and lower-case Latin indices indicate elements of an array. In general, Greek indices are associated with local coordinate systems and range from 1 to 2. The following symbols are used:

a_{ij}	constants in displacement approximation
$c_{N\alpha}$	node displacement coefficients
d_i	components of rigid-body translation
e	element identification index
n	total number of nodes in assembled system
p_{Nk}	generalized node forces in local coordinates
$\bar{p}_{Nk}$	generalized node forces in transformed coordinates
P_{Nk}	generalized node forces in global coordinates
q	pressure
$Q_i, \bar{Q}_i$	element node forces due to q
u_i	displacement components in local coordinates
u_{Ni}	element node displacements in local coordinates
$\bar{u}_{Ni}$	element node displacements in transformed coordinates
v, v_0	volume of deformed and undeformed element
x_i	local coordinates
x_{eNi}	coordinates of node N of element e
y_i	local coordinates of deformed element
A, A_0	area of deformed and undeformed element
C, C_1, C_2	material constants
E	total number of finite elements
E_{ijkl}	array of material constants
I_1, I_2, I_3	strain invariants
Z_i	global coordinates

Z_{Ni}	global coordinates of node N
β_{ij}	transformation matrix
γ_{ij}	Lagrangian strain tensor
δ_{ij}	Kronecker delta
λ	extension ratio
$\sigma_{\alpha\beta}$	stress tensor
Ω_{eMN}	homomorphic mapping constants

INTRODUCTION

THE general theory of elastic membranes can be regarded as a special case of the general theory of finite elasticity. In many respects, it is a simpler theory since it treats the stress field as essentially two-dimensional. Moreover, in the case of incompressible materials, the hydrostatic pressure can be determined immediately from the condition that the stress normal to the deformed membrane surface becomes negligible in comparison with the stresses in the tangent plane as the thickness of the membrane diminishes. Practically speaking, however, quantitative descriptions of the behavior of thin membranes are often more complex than that of three-dimensional bodies. Whereas the behavior of many structural systems are adequately described by linear theory, the behavior of most membranes is inherently nonlinear. Lateral pressure, for example, can stretch an initially flat membrane into a spheroidal balloon; then elements of the membrane undergo large displacements and rotations, and the strains of the middle surface may be appreciably greater than unity. For such problems, the theory of elastic membranes leads to systems of differential equations which, even in the case of relatively simple loading and geometry, must be integrated by numerical methods in order to obtain quantitative results.

The theory of elastic membranes, as for example, propounded by Green and Adkins [1], treats the problem as one of plane stress in a continuous media. Regardless of the concepts involved in the formulation, however, if numerical methods are employed to obtain quantitative results, the continuum is, in effect, approximated by a discrete model in the solution process. This paper investigates a logical alternative to this classical approach; namely, that of representing the continuum by a discrete model at the onset. The formulation of consistent discrete model is accomplished by use of the finite element concept [2] wherein a continuum is approximated by a finite number of component parts called elements. In this way, the problem reduces to one of determining a finite number of discrete variables which, in this study, are taken as the displacements of selected node points on the membrane.

The literature on applications of the finite element technique to general nonlinear problems is not extensive. The first application of this method to relatively simple geometrically nonlinear problems appears to have been presented by Turner *et al.* [3]. Subsequent contributions were made by Argyris [4] and Martin [5], among others. Nonlinear finite element formulations of certain geometrically nonlinear elasticity problems were presented by Wissmann [6, 7], whose work is based on the assumption of small strains and an ideal Hookean material. Wissmann's approach has the advantage that nonlinear characteristics in the behavior of a system can be identified in the development and the relative importance of nonlinear terms can be easily evaluated. The majority of the literature available on the subject, however, including the original work [3], takes into account geometric nonlinearity by successive corrections to a linearized problem. The paper by

Martin [5] contains a survey of the literature using this successive correction approach. No applications to finite strains appear to be available in the literature.

The present paper adopts Wissmann's approach in that the formulation leads to nonlinear stiffness relations. In this case, however, these relations are valid for gross deformations of membranes of general shape under variable pressures, and account for the effects of finite strains.

The discussion to follow is divided into six parts. In the first of these, various kinematic and geometric properties of the discrete model a membrane are examined, and it is shown that a continuous membrane can be represented by an assembly of a finite number of small triangular elements connected together at their vertices. The displacement fields within these elements are assumed to be linear functions of the initial coordinates and, as such, they can be expressed in terms of the displacements of the element's nodes. This discussion is followed by a section on strains in continuous membranes wherein the strain tensor and its invariants are expressed in terms of the generalized displacements. With this done, the kinematics of the discrete system are completely defined. Next, the principle of virtual work is used to derive general nonlinear stiffness relations for finite elements of various materials. Particular emphasis is given to the analysis of homogeneous isotropic membranes constructed of incompressible materials which exhibit strain energy functions of the Mooney [8] or the neo-Hookean [9] form. The analysis of Hookean membranes undergoing small strains but large displacements is also examined. It is shown that in the case of large deformations stiffnesses no longer transform congruently, as in the linear case. The fourth section contains a discussion of transformations which assemble the discrete elements into a single unit, and the fifth section deals with the analysis of membranes subjected to external pressure. The final section contains numerical results and discussions of convergence and the numerical procedure used to solve the systems of nonlinear equations generated in the analysis.

KINEMATIC CONSIDERATIONS

Consider a homogeneous elastic membrane which, in its initial (unstrained) state occupies a finite region $\mathscr{R}_0$ in space. The region $\mathscr{R}_0$ is defined by two material surfaces which are a uniform distance h_0 apart; the parallel surface bisecting h_0 is the middle surface of the membrane. The locations of points in $\mathscr{R}_0$ are specified by an orthogonal Cartesian coordinate system Z_i whose origin is not necessarily within the region. The Z_i coordinates are referred to as the global coordinates of the system.

Since an infinite number of material points lie in $\mathscr{R}_0$, an infinity of coordinates Z_i are required to completely specify the initial configuration of the membrane. To reduce this continuously distributed system to a discrete one, the membrane is approximated by a discrete model consisting of a finite number E of flat triangular elements. The geometry of each element in the initial state is thus defined by a plane of uniform thickness h_0 bounded by straight lines which intersect at three points, called the node points of the element. Let n denote the total number of nodes in the assembled system and let Z_{Ni} ($N = 1, 2, \ldots, n$; $i = 1, 2, 3$) denote the global coordinates of a typical node N. Then the set of numbers Z_{Ni} constitutes a finite point set which defines the geometry of the discrete system. The set Z_{Ni} is said to describe the external space of the system.

In addition to the global reference frame, fixed local coordinate systems x_{ei}($e = 1, 2, \ldots E$; $i = 1, 2, 3$) are established in the neighborhoods of each element. The x_{ei} are referr

to as the local coordinates of element e. The behavior of each element is to be first described independently in terms of its local coordinates and is then to be transformed into global coordinates when the connectivity and the boundary conditions are established. For simplicity, it is assumed that each element e lies in the x_{e1}, x_{e2} plane of its local coordinate system.

A rigid rotation of a typical local system x_{ei} transforms it into the $\bar{x}_{ei}$ system whose coordinate lines are parallel to the corresponding global coordinate lines. That is, x_{ei} are rotated parallel to the global coordinates Z_i by the orthogonal transformation

$$\bar{x}_{ei} = \beta_{eji} x_{ej} \quad \text{(no sum on } e\text{)} \tag{1}$$

where β_{eji} is the direction cosine of the angle between Z_i (or $\bar{x}_{ei}$) and x_{ej}. Clearly, $\bar{x}_{ei}$ and Z_i differ only by a constant vector.

Following a procedure similar to that used for global coordinates, the local coordinates of a typical node N of element e are denoted x_{eNi} ($e = 1, 2, \ldots, E$; $N, i = 1, 2, 3$); or, if the local coordinates are rotated parallel to the global coordinates, the local coordinates of N are denoted $\bar{x}_{eNi}$. Further, a numbering sequence of the nodes associated with each element is adopted whereby once node 1 of an element is identified, nodes 2 and 3 follow in cyclic order according to the right-hand rule. Note that due to the particular choice of local coordinates, $x_{eN3} = 0$.

The set of numbers $\bar{x}_{eNi}$ (or x_{eNi}) constitutes a finite point set which is said to describe the internal space associated with element e. The transformation which maps points in the external space (that is, points in the set Z_{Ni}) into points in the internal spaces (that is, points in the sets $\bar{x}_{eNi}$) establishes the connectivity of the system and, in effect, assembles the individual elements into a single discrete system. Such transformations are to be discussed later. The discrete model of the membrane and the local and global coordinates are illustrated in Fig. 1.

Attention is now confined to a typical element of the system and the element index e is temporarily dropped for clarity.

A general motion of the membrane is now considered which carries the system from its initial configuration $\mathscr{R}_0$ to a deformed configuration $\mathscr{R}$. In general, straight lines connecting node points in $\mathscr{R}_0$ will become curved lines in $\mathscr{R}$. However, if the node points in $\mathscr{R}_0$ are selected sufficiently close to one another, node lines in the deformed body are closely approximated by a system of straight line segments. This is equivalent to assuming that the displacement fields corresponding to a given element are linear functions of the local coordinates of that element. That is,

$$u_i = d_i + a_{i\alpha} x_\alpha \qquad (i = 1, 2, 3; \alpha = 1, 2) \tag{2}$$

where u_i are the components of displacement, d_i are the rigid-body translations of the element, and $a_{i\alpha}$ are undetermined constants.

Evaluating equation (2) at each of the three node points gives the set of relations

$$u_{Ni} = d_{Ni} + a_{i\alpha} x_{N\alpha} \tag{3}$$

in which u_{Ni} ($N = 1, 2, 3$) is the displacement of node N in direction i, d_{Ni} is the part of u_{Ni} due to the rigid translation of the element, and $x_{N\alpha}$ ($\alpha = 1, 2$) are the local coordinates of node N. Again, it is understood that indices $N = 1, 2, 3$ are to be associated with the three nodes of element e. The components d_{Ni}, of course, are the same for all nodes of the element.

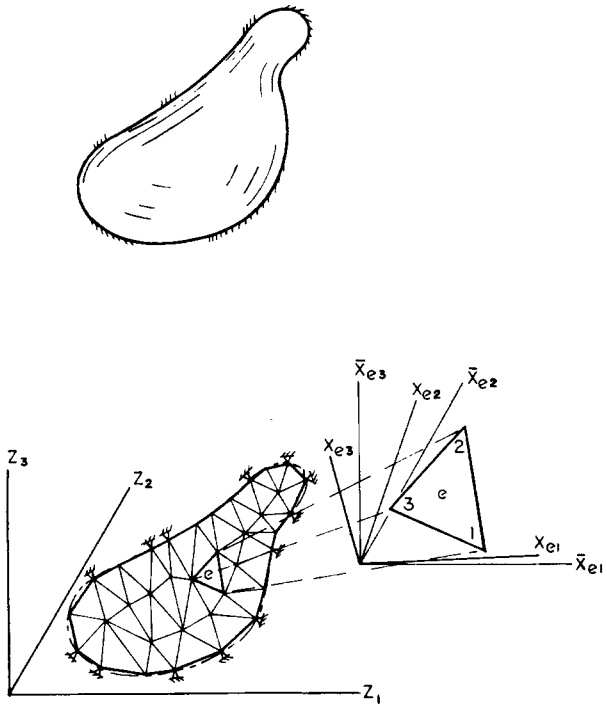


FIG. 1. Finite element representation of a membrane.

Thus, by setting

$$\begin{aligned} d_{11} &= d_{21} = d_{31} = d_1 \\ d_{12} &= d_{22} = d_{32} = d_2 \\ d_{13} &= d_{23} = d_{33} = d_3 \end{aligned} \tag{4}$$

equations (3) reduce to nine simultaneous equations for the three components d_i and the six parameters $a_{i\alpha}$. If equations (3) are expanded by first letting the index i take on values of 1, 2, and 3 and then expanding on N, the form of the coefficient matrix is particularly easy to invert. It is found that the solutions are

$$d_i = k_N u_{Ni} \tag{5}$$

and

$$a_{i\alpha} = c_{N\alpha} u_{Ni} \tag{6}$$

where

$$\begin{aligned} k_1 &= (x_{21}x_{32} - x_{31}x_{22})/2A_0 \\ k_2 &= (x_{12}x_{31} - x_{11}x_{32})/2A_0 \\ k_3 &= (x_{11}x_{22} - x_{12}x_{21})/2A_0 \end{aligned} \tag{7}$$

and

$$c_{N\alpha} = \frac{1}{2A_0}\begin{bmatrix} x_{22}-x_{32} & x_{31}-x_{21} \\ x_{32}-x_{12} & x_{11}-x_{31} \\ x_{12}-x_{22} & x_{21}-x_{11} \end{bmatrix} \tag{8}$$

Here A_0 is the area of the undeformed triangle. The quantities $c_{N\alpha}$ are called the displacement coefficients. Note that they are independent of the deformation of the membrane and are computed directly from the geometry of the undeformed element.

Introducing equations (5) and (6) into equation (2) gives

$$u_i = k_N u_{Ni} + c_{N\alpha} u_{Ni} x_\alpha \tag{9}$$

Similarly, the derivatives of the displacement components are given by

$$\frac{\partial u_i}{\partial x_\alpha} = c_{N\alpha} u_{Ni} \tag{10}$$

MEMBRANE STRAIN TENSOR

In order to describe the deformation of a continuous body, the region $\mathscr{R}_0$ occupied by the body at some reference time τ_0 is identified and the configuration of the body in this initial state is specified by the metric coefficients of material lines within $\mathscr{R}_0$. In the present case, these material lines are assumed to be initially parallel to the fixed Cartesian system x_i so that the metric tensor at τ_0 is simply the Kronecker delta, δ_{ij}. The motion of the body is assumed to carry it continuously from one configuration to another so that at some time $\tau > \tau_0$ the body occupies the region $\mathscr{R}$. The location of points originally given by coordinates x_i now have coordinates

$$y_i = x_i + u_i \tag{11}$$

with respect to the fixed local system x_i.

According to Green and Adkins [1], the Lagrangian strain tensor is defined by the relation

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j} - \delta_{ij}\right) \tag{12}$$

where δ_{ij} is the Kronecker delta. For very thin membranes, the deformation is assumed to be symmetric about the middle surface $y_3 = 0$ so that y_3 is an even function of x_3. It follows that on the middle surface

$$\left.\frac{\partial y_3}{\partial x_3}\right|_{x_3=0} = \lambda(x_1, x_2) \tag{13}$$

where λ is a scalar function representing the extension ratio at the middle surface in a direction normal to the middle surface.

In the following developments, it is assumed that the membrane thickness is so small in comparison with other characteristics dimensions that the strain components are essentially uniform over h. If such is the case,

$$\lambda = \frac{h}{h_0} \tag{14}$$

where h_0 and h are the membrane thicknesses in $\mathscr{R}_0$ and $\mathscr{R}$ respectively. It then follows from equations (12), (13), and (14) that for thin membranes

$$\begin{aligned} \gamma_{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_i}{\partial x_\beta}\right) \\ \gamma_{\alpha 3} &= 0 \\ \gamma_{33} &= \tfrac{1}{2}(\lambda^2 - 1) \end{aligned} \tag{15}$$

in which $\alpha, \beta = 1, 2$ and $i = 1, 2, 3$. Further, in this case the three invariants of the tensor γ_{ij} can be expressed in the form

$$\begin{aligned} I_1 &= \lambda^2 + 2(1 + \gamma_{\alpha\alpha}) \\ I_2 &= -\lambda^4 + \lambda^2 I_1 + \frac{1}{\lambda^2} I_3 \\ I_3 &= \lambda^2(1 + 2\gamma_{\alpha\alpha} + 2e_{\alpha\beta}e_{\lambda\mu}\gamma_{\alpha\lambda}\gamma_{\beta\mu}) \end{aligned} \tag{16}$$

where $e_{\alpha\beta}$ and $e_{\lambda\mu}$ are the two-dimensional permutation symbols ($e_{12} = 1$, $e_{21} = -1$, $e_{11} = e_{22} = 0$).

For the discrete system, the strain components are expressed in terms of the node displacements by introducing equation (10) into equations (15):

$$\begin{aligned} \gamma_{\alpha\beta} &= \tfrac{1}{2}(c_{N\alpha}u_{N\beta} + c_{N\beta}u_{N\alpha} + c_{N\alpha}c_{M\beta}u_{Ni}u_{Mi}) \\ \gamma_{\alpha 3} &= 0 \\ \gamma_{33} &= \tfrac{1}{2}(\lambda^2 - 1) \end{aligned} \tag{17}$$

wherein $\alpha, \beta = 1, 2$ and $M, N, i = 1, 2, 3$.

Note that equations (17) give the strains in terms of a specified displacement field. Hence, the compatibility equations are automatically satisfied within the boundaries of each finite element.

NONLINEAR STIFFNESS RELATIONSHIP

Let p_{Mk} denote the generalized force at node M acting in the direction k corresponding to the node displacement u_{Mk}. If the node displacements are given variations δu_{Mk}, the forces p_{Mk} perform an amount of external virtual work equal to $p_{Mk}\delta u_{Mk}$. Assuming that the deformation process is reversible and isothermal, an elastic potential function W exists which represents the strain energy per unit volume of the undeformed element. In general, W can be written as a function of the strains, and, by virtue of equations (17), it can also be expressed as a function of the node displacements. It then follows that the variations δu_{Mk}

lead to a variation in the strain energy per unit volume of

$$\delta W = \frac{\partial W}{\partial u_{Mk}} \delta u_{Mk} \tag{18}$$

The total internal virtual work is then

$$\int_v \delta W \, dv_0 = v_0 \frac{\partial W}{\partial u_{Mk}} \delta u_{Mk} \tag{19}$$

where v_0 is the volume of the undeformed element. Note that it is permissible to factor δW outside the integral since W is a function of strains which, in view of equations (17), are constant throughout the element.

From the principle of virtual work, the strained element reaches an equilibrium state when the internal and external virtual work are equal for arbitrary values of δu_{Mk}. Therefore,

$$p_{Mk} = v_0 \frac{\partial W}{\partial u_{Mk}} \tag{20}$$

The precise form of the stiffness relation in equation (20) depends upon the form of the elastic potential function W, which, in turn, depends upon the type of material of which the element is composed. In the following, it is assumed that the element is composed of an isotropic, perfectly elastic material, in which case

$$W = W(I_1, I_2, I_3) \tag{21}$$

where I_1, I_2, and I_3 are invariants of the strain tensor defined in equations (16).

The general stiffness relation for a finite element of an elastic body is obtained by introducing equation (21) into equation (20):

$$p_{Mk} = \frac{v_0}{2}\left[\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \gamma_{ij}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \gamma_{ij}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \gamma_{ij}}\right)\frac{\partial \gamma_{ij}}{\partial u_{Mk}} + \left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \gamma_{ji}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \gamma_{ji}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \gamma_{ji}}\right)\frac{\partial \gamma_{ji}}{\partial u_{Mk}}\right] \tag{22}$$

Note that

$$v = v_0\sqrt{I_3} \tag{23}$$

In differentiating the invariants with respect to γ_{ij} it is understood that all other strain components are held constant, including γ_{ji}.

Rivlin [10] noted that for compressible materials W can be expressed as a polynomial function of the strain invariants of the form

$$W = \sum_{r,s,t} B_{rst}(I_1 - 3)^r (I_2 - 3)^s (I_3 - 1)^t \; B_{000} = 0 \tag{24}$$

where B_{rst} are material constants. When written in this form, W is zero in the unstrained state of the body. Toupin and Bernstein [11] used an alternate but equivalent formula for W for compressible materials, which is obtained by representing W as a polynomial in the

strains of the form

$$W = \tfrac{1}{2}(E_{ijkl}\gamma_{ij}\gamma_{kl} + \tfrac{1}{3}E_{ijklmn}\gamma_{ij}\gamma_{kl}\gamma_{mn} + \ldots) \tag{25}$$

where E_{ijkl}, E_{ijklmn}, ..., etc. are multi-dimensional arrays of material constants. In the case of small strains, W is approximated by taking only the first term in the polynomial in equation (25) and W reduces to the energy function of the classical Hookean material:

$$W = \tfrac{1}{2}E_{ijkl}\gamma_{ij}\gamma_{kl} \tag{26}$$

In the case of incompressible materials, the third strain invariant I_3 equals unity, the volumes v_0 and v of the undeformed and the deformed elements are equal, and W is given only in terms of I_1, and I_2.

For certain incompressible rubber-like materials, Mooney [8] has proposed a strain-energy function of the form

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{27}$$

where C_1 and C_2 are experimentally determined constants. Rivlin and Saunders [12, 13] verified experimentally that equation (27) gives a suitable form of W for certain highly elastic materials such as vulcanized rubber. Elastic materials with strain–energy functions of this form are often referred to as Mooney–Rivlin materials.

Treloar [14], using a statistical approach based on the molecular theory of highly-elastic materials, found for incompressible materials

$$W = C(I_1 - 3) \tag{28}$$

where C is a constant. Rivlin [10] refers to materials exhibiting this form of strain–energy function as neo-Hookean.

Hereafter, attention is confined to elastic membranes constructed of incompressible materials with strain–energy functions of the form given in equations (27) or (28) or compressible materials subjected to small strains, for which the strain–energy function is of the form indicated in equation (26).

Consider first an incompressible membrane of Mooney–Rivlin material. From the incompressibility condition $I_3 = 1$, it is found that

$$\begin{aligned} I_1 &= \lambda^2 + 2(1 + \gamma_{\alpha\alpha}) \\ I_2 &= \frac{1}{\lambda^2} + 2\lambda^2(1 + \gamma_{\alpha\alpha}) \\ \lambda^2 &= (1 + 2\gamma_{\alpha\alpha} + 2e_{\alpha\beta}e_{\lambda\mu}\gamma_{\alpha\lambda}\gamma_{\beta\mu})^{-1} \end{aligned} \tag{29}$$

Further note that

$$\begin{aligned} \frac{\partial\lambda^2}{\partial\gamma_{\alpha\beta}} &= -2\lambda^4 f_{\alpha\beta}\,; \qquad \frac{\partial(1/\lambda^2)}{\partial\gamma_{\alpha\beta}} = 2f_{\alpha\beta} \\ \frac{\partial I_1}{\partial\gamma_{\alpha\beta}} &= 2(\delta_{\alpha\beta} - \lambda^4 f_{\alpha\beta}) \\ \frac{\partial I_2}{\partial\gamma_{\alpha\beta}} &= 2f_{\alpha\beta}(1 - 2\lambda^4 - 2\lambda^4\gamma_{\lambda\lambda}) + 2\lambda^2\delta_{\alpha\beta} \end{aligned} \tag{30}$$

where

$$f_{\alpha\beta} = \delta_{\alpha\beta} + 2e_{\mu\alpha}e_{\lambda\beta}\gamma_{\mu\lambda} \tag{31}$$

The scalar λ^2 and the surface tensor $f_{\alpha\beta}$ are expressed in terms of the node displacements by introducing equations (17) into equations (29) and (31):

$$\begin{aligned}\lambda^2 &= [1 + c_{N\alpha}(u_{N\alpha} + \tfrac{1}{2}c_{M\alpha}u_{Ni}u_{Mi}) + e_{\alpha\beta}e_{\lambda\mu}(c_{N\alpha}u_{N\lambda} \\ &\quad + \tfrac{1}{2}c_{N\alpha}c_{M\lambda}u_{Ni}u_{Mi})(c_{L\beta}u_{L\mu} + c_{L\mu}u_{L\beta} + c_{K\beta}c_{L\mu}u_{Kj}u_{Lk})]^{-1} \\ f_{\alpha\beta} &= \delta_{\alpha\beta} + e_{\mu\alpha}e_{\lambda\beta}(c_{N\mu}u_{N\lambda} + c_{N\lambda}u_{N\lambda} + c_{N\lambda}c_{M\mu}u_{Ni}u_{Mi}) \\ &\quad i, j, L, M, K = 1, 2, 3; \alpha, \beta, \lambda, \mu, = 1, 2\end{aligned} \tag{32}$$

Finally, introducing equations (29) through (32) into equation (22) and simplifying gives the nonlinear stiffness relation for a finite element of a Mooney–Rivlin membrane:

$$\begin{aligned}p_{Nk} &= 2v_0c_{N\alpha}(\delta_{\beta k} + c_{M\beta}u_{Mk})\{C_1(\delta_{\alpha\beta} - \lambda^4 f_{\alpha\beta}) \\ &\quad + C_2[f_{\alpha\beta}(1 - 2\lambda^4 - 2\lambda^4\gamma_{\mu\mu}) + \gamma^2\delta_{\alpha\beta}]\}\end{aligned} \tag{33}$$

In this equation,

$$\gamma_{\mu\mu} = c_{N\mu}(u_{N\mu} + \tfrac{1}{2}c_{M\mu}u_{Ni}u_{Mi}) \tag{34}$$

and λ^2 and $f_{\alpha\beta}$ are as given in equations (32).

The stresses in the element are calculated by means of the formula [1]

$$\sigma_{\alpha\beta} = 2\lambda\{C_1(\delta_{\alpha\beta} - \lambda^4 f_{\alpha\beta}) + C_2[\lambda^2\delta_{\alpha\beta} + (1 - 2\lambda^4 - 2\lambda^4\gamma_{\mu\mu})f_{\alpha\beta}]\} \tag{35}$$

where $\sigma_{\alpha\beta}$ is the stress tensor per unit of cross-sectional area of the deformed membrane.

The nonlinear stiffness relation for neo-Hookean membranes is obtained directly from equation (33) by replacing C_1 by C and equating C_2 to zero:

$$p_{Nk} = 2v_0c_{N\alpha}C(\delta_{\beta k} + c_{M\beta}u_{Mk})(\delta_{\alpha\beta} - \lambda^4 f_{\alpha\beta}) \tag{36}$$

Similarly, for neo-Hookean membranes, equation (35) reduces to

$$\sigma_{\alpha\beta} = 2\lambda C(\delta_{\alpha\beta} - \lambda^4 f_{\alpha\beta}) \tag{37}$$

In the case of Hookean membranes, strains are assumed to be small in comparison with unity and the volume of the deformed element is approximately the same as that of the undeformed element. Noting that for an isotropic membrane the tensor E_{ijkl} has the symmetries

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij}$$

and introducing equations (17) and (26) into equation (22) gives

$$\begin{aligned}p_{Nk} &= v_0c_{N\alpha}(\delta_{\beta k} + c_{M\beta}u_{Mk})E_{\alpha\beta\lambda\mu}c_{I\lambda}(\delta_{\mu i} + \tfrac{1}{2}c_{J\mu}u_{Ji})u_{Ii} \\ &\quad \alpha, \beta, \lambda, \mu = 1, 2; I, J, M, N, i, k = 1, 2, 3\end{aligned} \tag{38}$$

This result agrees with that obtained by Wissman [6].

Stresses in Hookean membranes are obtained from the formula

$$\sigma_{\alpha\beta} = E_{\alpha\beta\lambda\mu}c_{N\lambda}(u_{N\mu} + \tfrac{1}{2}c_{M\mu}u_{Mi}u_{Ni}) \tag{39}$$

If products of the node displacements are neglected in comparison with the displacements themselves, equation (38) reduces to the linear stiffness relation for plane stress:

$$p_{N\alpha} = v_0 c_{N\beta} E_{\alpha\beta\lambda\mu} c_{I\lambda} u_{I\mu} \tag{40}$$

The array $E_{\alpha\beta\lambda\mu}$ can be interpreted as the stiffness matrix for an infinitesimal volume of the membrane. The remaining terms in equation (40) then transform this matrix congruently into one associated with the finite element. On comparing equations (40) and (38), it is seen that, contrary to the linear theory, congruent transformations are not admissible in the case of large displacements.

MEMBRANE ANALYSIS

The nonlinear stiffness relations derived in the previous section describe the behavior of a single finite membrane element within its local reference frame; these relations are independent of the loading on the membrane, the boundary conditions, or the location of the element in the assembled system. It is now necessary to connect the elements and to sum their properties so as to represent a membrane of specified shape with specified boundary conditions. Toward this end, it is convenient to first rotate the node forces, node displacements and local coordinates associated with each element so that they are parallel to the global reference frame Z_i. This is accomplished through the transformations

$$\begin{aligned} \bar{p}_{eMi} &= \beta_{eki} p_{eMk} \\ u_{eMk} &= \beta_{eki} \bar{u}_{eMi} \\ x_{ei} &= \beta_{eki} x_{ek} \qquad \text{(no sum on } e\text{)} \end{aligned} \tag{41}$$

where $\bar{p}_{eMi}$ and $\bar{u}_{eMi}$ are the node forces and displacements of node M of element e in the direction Z_i, $\bar{x}_{ei}$ are the rotated local coordinates defined in equation (1) and β_{eki} is the cosine of the angle between $\bar{x}_{ei}$ and x_{ek}. In these equations, $M, i, k = 1, 2, 3$ and $e = 1, 2, \ldots, E$ where E is the total number of finite elements.

It was pointed out earlier that the set of numbers $Z_{Ni} (i = 1, 2, 3;\ N = 1, 2, \ldots, n)$ describes the geometry of the assembled (connected) system whereas $\bar{x}_{eNi} (e = 1, 2, \ldots, E;\ N, i = 1, 2, 3)$ describes that of the individual elements. The first set is referred to as the external space of the discrete system and the second set is referred to as the internal space. The connectivity of the system is established by relating the members of the set Z_{Ni} to those of $\bar{x}_{eNi}$ by the transformation

$$\bar{x}_{eMi} = \Omega_{eMN} Z_{Ni} \ (M, i = 1, 2, 3;\ D = 1, 2, \ldots, n) \tag{42}$$

where

$$\Omega_{eMN} = \begin{cases} 1 & \text{if node } M \text{ of the element } e \text{ is identical to node } N \text{ in the assembled system, and} \\ 0 & \text{if otherwise} \end{cases} \tag{43}$$

The transformation indicated in equation (42) defines a homomorphic mapping of points in Z_{Ni} into points in $\bar{x}_{eNi}$ and, in effect, assembles the elements into a single unit.

Similarly, if P_{Nk} and U_{Nk} are the node forces and displacements of the assembled system (external space), it follows that

$$\begin{aligned} P_{Nk} &= \Omega_{eMN}\bar{p}_{eMk} \\ \bar{u}_{eMk} &= \Omega_{eMN}U_{Nk} \end{aligned} \tag{44}$$

In this case the repeated index e is summed throughout its entire range: $e = 1, 2, \ldots, E$.

Application of equations (41) through (44) completes the assembly of the discrete system. To demonstrate, let p_{Mk} and u_{Rn} denote the node forces and displacements corresponding to a finite element e. Then a nonlinear stiffness relation can be found of the form

$$p_{eMm} = k(u_{eNk}) \tag{45}$$

where $k(u_{eNk})$ is the appropriate nonlinear function of the node displacements. The functions $k(u_{eNk})$ for Mooney–Rivlin, neo-Hookean, and Hookean materials are defined by equations (33), (36), and (38) respectively.

When the forces and displacements are rotated so that they are parallel to coordinates $\bar{x}_{ei}$, equation (45) becomes

$$\bar{p}_{eMi} = \bar{K}(\bar{u}_{eNj}) \tag{46}$$

where $\bar{p}_{eMi}$ and $\bar{u}_{eNj}$ are defined in equations (41) and

$$\bar{K}(\bar{u}_{eNj}) = \beta_{eim}k(\beta_{\underline{e}kj}\bar{u}_{\underline{e}Nj}) \tag{47}$$

The underscored indices in this equation are not summed.

Finally, the global stiffness relations relate external node forces to node displacements of the assembled system:

$$P_{Ir} = K(U_{Js}) \tag{48}$$

where P_{Ir} and U_{Js} are given by equations (41) and

$$K(U_{Js}) = \Omega_{eMI}\beta_{eim}k(\beta_{\underline{e}kj}\Omega_{\underline{e}NJ}U_{Js})$$

Boundary conditions are applied by prescribing generalized forces or displacements at the appropriate boundary nodes. Then equations (48) reduce to a system of independent nonlinear algebraic equations in the unknown node displacements. This completes the formulation of the problem.

EXTERNAL PRESSURE

As they now stand, the above stiffness relations are applicable only to membranes subjected to loads which do not change in direction as the membrane deforms. Since this is obviously a severe restriction in membrane analysis, attention is now directed toward the adaptation of the method to account for changes in the external loading due to deformation.

Consider an elastic membrane subjected to a nonuniform lateral pressure. As the membrane deforms, not only does the direction of this pressure change but also the area on which it acts changes. To account for this, it is first assumed that the dimensions of each finite element are sufficiently small that the pressure is essentially uniform over the surfaces

of individual elements. Then, if q denotes the applied pressure on a typical element and A denotes the surface area of the deformed element, the total force exerted normal to the plane of the element is clearly

$$\tilde{q} = Aq \tag{49}$$

Now let v_i denote the components of an outward unit vector normal to A. Then the components of the pressure force $\tilde{q}$ are given by

$$\tilde{q}_i = v_i Aq \tag{50}$$

To determine the components v_i, note that according to equation (11) the coordinates of nodes of the deformed element are

$$y_{Ni} = x_{Ni} + u_{Ni} \tag{51}$$

For convenience in writing, the origin of the reference frame y_i is translated into node point 3 of the deformed element. If the resulting coordinate system is denoted z_i, it follows that

$$z_{Ni} = y_{Ni} - y_{3i} \tag{52}$$

Now consider two unit vectors **a** and **b** emanating from the origin of the coordinates z_i (node 3). The components v_i of the unit normal are obtained by forming the vector product of **a** and **b**:

$$v_i = \frac{1}{2A} e_{ijk} z_{1k} z_{2k} \tag{53}$$

where e_{ijk} is the permutation symbol. Thus, equation (50) can be written

$$q_i = \tfrac{1}{2} q e_{ijk} z_{1j} z_{2k} \tag{54}$$

The net external force at each node is obtained by simply representing q by three forces, one at each node, whose components are

$$Q_i = \tfrac{1}{6} q e_{ijk} z_{1j} z_{2k} \tag{55}$$

By introducing equation (52), equation (55) is transformed into the y_i coordinates:

$$Q_i = \tfrac{1}{6} q e_{ijk} (y_{1j} y_{2k} + y_{2j} y_{3k} + y_{3k} y_{1k}) \tag{56}$$

Note that no node identification index is needed since Q_i is the same for each node of the element.

Equation (56) defines the generalized external force in the deformed coordinates produced by pressure loading. To complete the analysis, it is now necessary to transform these forces into components $\bar{Q}_i$ parallel to the local reference frame $\bar{x}_i$. This is accomplished by introducing equation (51) into equation (56) and then rotating the x_i coordinates into the $\bar{x}_i$ system with the aid of equation (1). The result for element e is

$$\bar{Q}_{ei} = \tfrac{1}{12} q_e \sum_{p=1}^{3} \delta_{ijk}^{pMN} (\bar{x}_{eNj} + \bar{u}_{eNj})(\bar{x}_{eMk} + \bar{u}_{eMk}) \quad \text{(no sum on } e\text{)} \tag{57}$$

wherein δ_{ijk}^{pMN} is the generalized Kronecker delta ($\delta_{ijk}^{pMN} = e_{pMN} e_{ijk}$).

Equation (57) represents the final formula for the node forces on element e due to a uniform pressure over that element. In the case of pressure loadings, the components $\bar{Q}_{ei}$ take the place of the node forces $\bar{p}_{eNi}$ of equations (41) and (46). It is seldom necessary to transform these components into the global system, however, since it is more convenient to first transform displacements into the $\bar{x}_i$ system with the aid of equations (41) and (44) and then to transform the resulting forces into the global system.

NUMERICAL RESULTS

The systems of equations obtained in the present finite-element formulation are highly nonlinear in the unknown node displacements. Of the variety of numerical schemes available to solve systems of equations of this type, the Newton–Raphson method [15] is

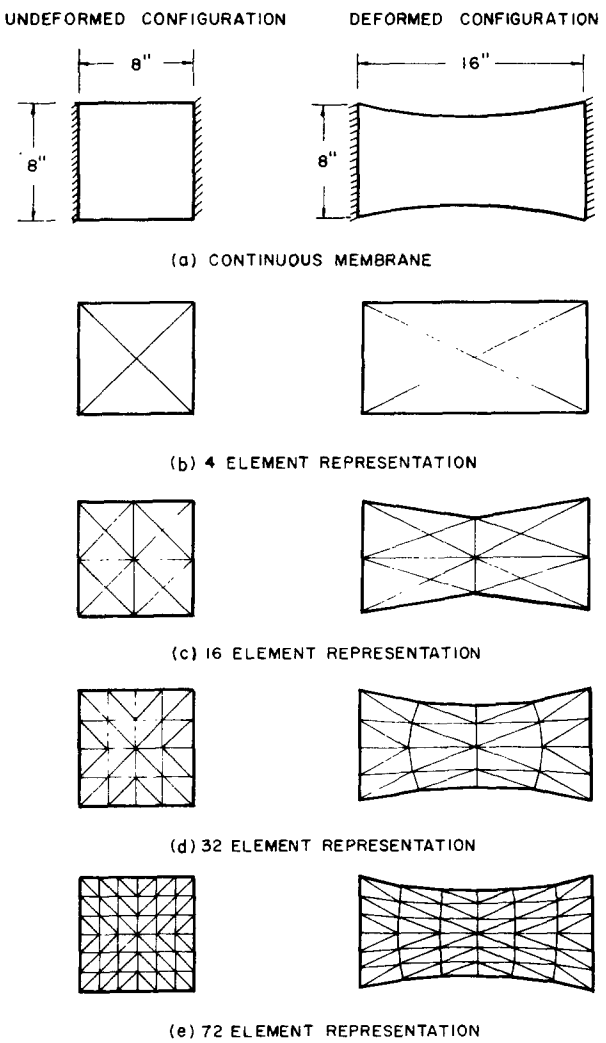


FIG. 2. Displaced configurations for various geometric representations for example A.

perhaps the most straight forward. This method was used in the present study to solve the nonlinear equations associated with finite element representations of several membrane structures.

Briefly, in the Newton–Raphson method, the nonlinear stiffness relation is expanded in a Taylor's series and truncated to only two terms. The first term represents a linear stiffness relation and the second term represents a correction due to increments in the node displacements. The linearized equations are solved for the displacements produced by increments in the node forces and these are introduced into the nonlinear equations to obtain corrected values for the force increments. A new set of linearized equations are then computed using the truncated Taylor series. By successively solving the corrected linearized equations for displacements produced by the corrected forces of the previous cycle and correcting the node forces each time using the nonlinear equations, an iterative scheme is established which can be used to solve large systems of highly nonlinear algebraic equations.

In cases of concentrated loads and re-entrant corners, the numerical procedure converges for any given finite element representation. However, as the finite element network is refined, stresses in the neighborhoods of concentrated loads and re-entrant corners diverge, in agreement with the exact solutions. Moreover, no provisions have been incorporated in the analysis to seek out adjacent equilibrium states for given loading conditions. As a result, in some applications of the technique, ill-conditioned stiffness matrices may arise from an improper choice of starting values of the undetermined displacements. This

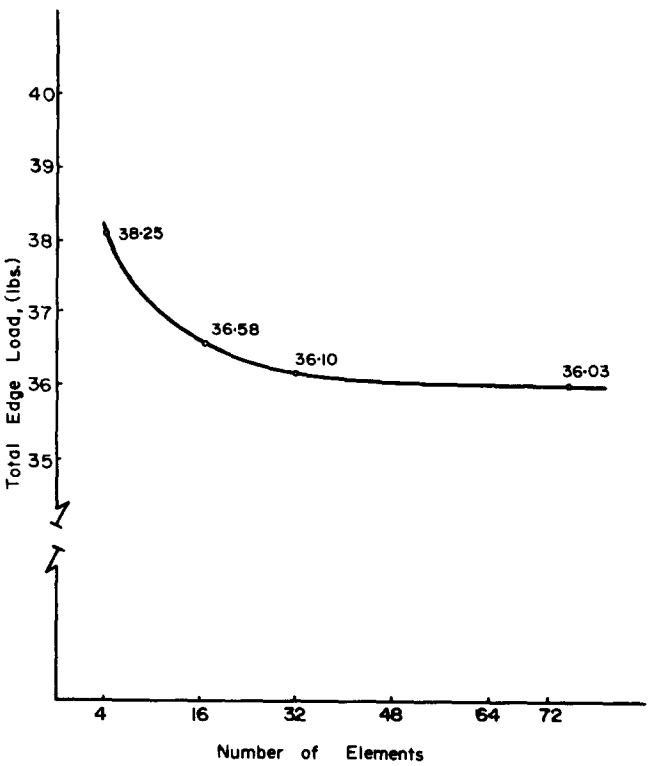
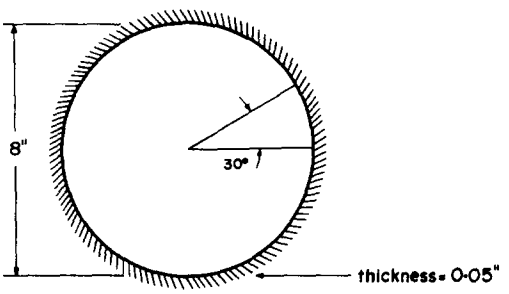


FIG. 3. Total edge load versus number of elements for stretching of a square membrane.

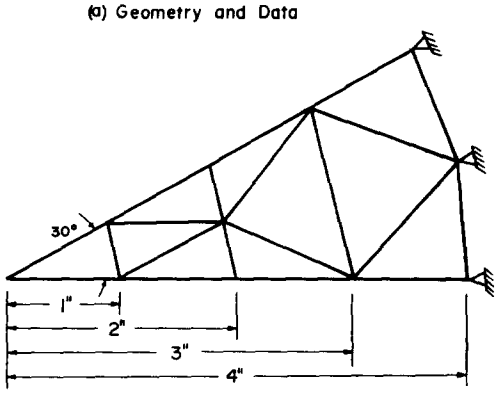
can often be attributed to the fact that single-valued displacement fields seldom exist when portions of the membrane are subjected to high compressive stresses. In such cases, negative diagonal elements may be encountered in the linearized incremental stiffness matrices. These matrices are then no longer positive definite and the iteration scheme diverges. This situation can often be circumvented by arbitrarily prescribing a new set of initial displacements.

The rate of convergence of the Newton–Raphson method depends, of course, on the choice of initial values of the generalized displacements. As the number of elements is increased, the rate of convergence decreases. Computational experience indicates that in plane problems, such as that indicated in Fig. 2, convergence rates are considerably higher than in cases involving lateral pressure. In the analysis of membranes which undergo large out-of-plane deformations due to the action of prescribed external forces, rates of convergence can be significantly increased by first analyzing a very coarse finite element representation of the membrane using a small number of iterations. These results are then used as starting values for a more refined representation, the displacements of the added node points being obtained through linear interpolation.

Two examples are presented which indicate the application of the theory to problems involving large strains. The first example consists of a square membrane stretched in one direction to twice its original length. This is accomplished by prescribing displacements at nodes along one edge and fixing the nodes of the opposite edge, as indicated in Fig. 2. The purpose of this example is to verify the convergence of the solution process.



(a) Geometry and Data



(b) Typical 30° Segment

FIG. 4. Geometry and finite element representation for initially flat circular membrane.

It is assumed that the membrane is constructed of a Mooney–Rivlin material with material constants C_1 and C_2 of 4·0 and 1·5 psi respectively. These values correspond to those obtained experimentally by Rivlin and Saunders [12, 13] for synthetic rubber. Four different cases were computed to examine the effect of approximating the membrane by various finite element representations. The select patterns and the resulting configurations are shown in Fig. 3.

As an indication of the convergence of the method, the total horizontal edge force is plotted versus the number of finite elements in Fig. 3. This figure indicates that the edge force converges monotonically to approximately 36 pounds. In this case, a close approximation is obtained using a relatively crude finite element representation.

As a second example, the inflation of an initially flat circular membrane is considered. The membrane is composed of Mooney–Rivlin material with the same material constants as that of the first example and it is subjected to a uniform external pressure of 1·65 psi. Boundary conditions at the circumference are simulated by specifying zero displacement at 15° intervals in the discrete representation of the continuously supported membrane. The geometry and additional data for the example is given in Fig. 4.

A typical deformed segment and the computed displacement along a diameter for the discrete model are shown on the right side of Fig. 5. The left side of this figure depicts a conceptual configuration for a continuous membrane under the same loading obtained by fitting smooth curve through the displaced nodes. The profile of the deformed membrane is in good agreement with that given by Green and Adkins [1].

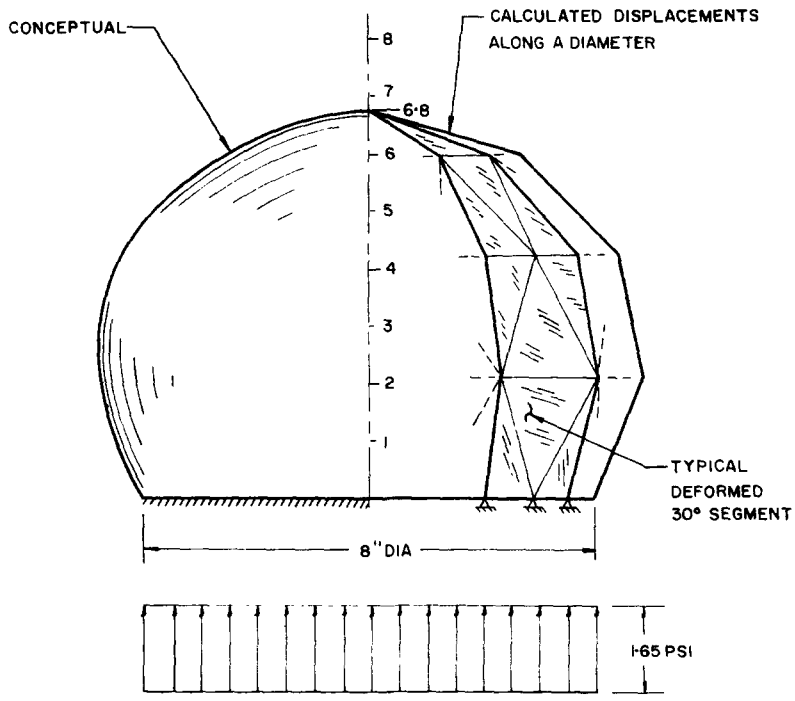


FIG. 5. Displaced configuration of an initially flat synthetic rubber membrane due to uniform pressure.

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Résumé—Cet exposé présente une formulation d'éléments finies consistente pour l'analyse de déplacements et de tensions finies dans les membranes élastiques d'une forme générale. Une membrane continue est divisée en un nombre d'éléments triangulaires plats et le comportement d'un élément typique est décrit relativement aux déplacements de ses noeuds. Il est supposé que les noeuds sont assez proches pour que les champs de déplacement dans chaque élément puissent être calculés approximativement au moyen des fonctions linéaires des coordonnées locales. A partir de cette supposition, l'élément tensoriel de déformation de Lagrange est exprimé par rapport aux déplacements des noeuds et une relation de rigidité non linéaire entre les forces et les deplacements des noeuds en est tirée. Des transformations de groupes sont introduites qui reassemblent les éléments et appliquent des conditions de limite appropriées. Celles-ci mènent à des systèmes d'équations algébriques non linéaires dans les déplacements généralisés. Des exemples numériques sont inclus pour démontrer le procédé.

Zusammenfassung—Diese Arbeit gibt eine konsistente endliche Formulierung zur Analyse grosser Verschiebungen und endlicher Dehnungen elastischer Membranen allgemeiner Form. Eine durchlaufende Membran wird in eine Anzahl flacher Dreieckselemente geteilt und das Verhalten eines typischen Elementes wird beschrieben, ausgedrückt als Verschiebungen der Knotenpunkte. Es wird vorausgesetzt, dass die Knotenpunkte so nahe zu den Verschiebungsfeldern der einzelnen Elemente sind, dass diese durch Linearfunktionen der lokalen Koordinaten angenähert werden können. Auf diese Voraussetzung gestützt ist der Lagrange'sche Spannungstensor als Knotenpunktverdrängung ausgedrückt und man erhält ein nichtlineares Verhältnis für die Knotenpunktkräfte und Verdrängungen. Gruppen Verwandlungen werden eingeführt, die die Elemente umordnen und entsprechende Grenzbedingungen einführen. Dies führt zu nichtlinearen algebraischen Gleichungssystemen der verallgemeinerten Verschiebungen. Numerische Beispiele werden gegeben um den Vorgang zu zeigen.

Абстракт—Работа представляет компактную конечную формулировку расчета больших перемещений и конечных деформаций упругой мембраны произвольной формы. Сплошная мембрана разделена нанекоторые число плоских трехугольных элементов, причем поведение типичного элемента выражено в зависимости от перемещений их узлов. Подразумивается, что узловые точки тесно связаны так, что поля перемещений внутри каждого элемента можна приблизить с помощью функций локальных координат. Исходя из этого предположения можна выразить тензор деформации Лагранжа в виде узлов перемещений. Применяется нелинейная зависимость для жесткости между узловыми силами и перемещениями. Приводятся групповые преобразования, которые собирают снова все элементы и удовлетворают точным граничным условиям. Это приводит к системе нелинейных алгебраических уравнений в обобщенных перемещениях. Даются численные примеры для облегчения расчетов.